

# Anomalous mean-field behavior of the fully connected Ising model

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## Abstract

Although the fully connected Ising model does not have a length scale, its critical exponents for thermodynamic quantities such as the mean magnetization and the susceptibility can be obtained using finite size scaling with the scaling variable equal to  $N$ , the number of spins. We show that at the critical temperature of the infinite system the mean value and the most probable value of the magnetization scale differently with  $N$ , and the magnetization probability distribution is not a Gaussian, even for large  $N$ . Similar results inconsistent with the usual understanding of mean-field theory are found at the spinodal. These results imply that non-thermodynamic quantities are not correctly obtained by a straightforward application of finite size scaling. We relate these results to the breakdown of hyperscaling and show how hyperscaling can be restored by increasing  $N$  while holding the Ginzburg parameter rather than the temperature fixed.

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## I. INTRODUCTION

Mean-field approaches to phase transitions are useful for several reasons. Two of the most important are that they provide a simple way of understanding the nature of critical phenomena [1], and they are good approximations for systems with long-range interactions and for large molecules [2, 3]. Despite the work of Kac and collaborators [4], who defined the applicability of mean-field theories in a mathematically precise manner, mean-field approximations are still approached in different ways. These different approaches can be confusing because they can produce different results for the same system. A common approach is to assume that the probability distribution of the order parameter is a Gaussian. Another common approach is to consider a system at its upper critical dimension.

In this paper we investigate another often used way of understanding mean-field behavior in Ising systems and consider a system in which every spin interacts with every other spin. More precisely, the Hamiltonian of the fully connected Ising model is given by [5–8]

$$H = -J_N \sum_{i>j, j=1}^N \sigma_i \sigma_j - h \sum_{i=1}^N \sigma_i, \quad (1)$$

where  $\sigma_i = \pm 1$  and  $h$  represents the external magnetic field. The interaction strength  $J_N$  is rescaled so that the total interaction energy of a given spin remains the same as  $N$  is changed. We take

$$J_N = \frac{qJ}{N-1}, \quad (2)$$

with  $q = 4$ . This choice of  $q$  yields the mean-field critical temperature  $T_c = 4$ , the value of  $T_c$  for a square lattice for  $N \rightarrow \infty$ , where we have chosen units such that  $J/k = 1$ , with  $k$  equal to the Boltzmann constant. The fully connected Ising model is sometimes referred to as the “mean-field” [9], “infinitely coordinated” [10, 11], or “infinite range” [12] Ising model.

We will show that finite size scaling at the critical temperature of fully connected Ising model yields results that are inconsistent with both the assumption of a Gaussian probability distribution and several results at the upper critical dimension.

The standard approach to finite size scaling yields numerical values of the critical exponents by determining how various quantities change with the linear dimension  $L$  at the critical point of the infinite system [13–15]. The finite size scaling relations for the Ising

model with finite-range interactions include

$$\overline{m} \sim L^{-\beta/\nu} \quad (3)$$

$$\chi \sim L^{\gamma/\nu}, \quad (4)$$

where  $\overline{m} = \overline{|M|}/N$ ,  $|M|$  is the absolute value of the magnetization of the system, the overbar denotes the ensemble average,  $\chi$  is the susceptibility per spin,  $N$  is the number of spins, and  $\beta$ ,  $\gamma$ , and  $\nu$  are the usual critical exponents. The exponents at the mean-field critical point are given by

$$\gamma = 1, \quad \beta = 1/2, \quad \text{and} \quad \nu = 1/2, \quad (5)$$

which yields  $\overline{m} \sim L^{-1}$  and  $\chi \sim L^2$ .

Because the fully connected Ising model has no length scale, the linear dimension  $L$  is not defined. One simple way to determine how  $\overline{m}$  and  $\chi$  change with  $N$  at the critical temperature is to assume that its critical exponents are the same as the nearest-neighbor Ising model in four dimensions, the upper critical dimension [16]. Given this assumption we can write  $N \sim L^4$ , and hence [17]

$$\overline{m} \sim N^{-1/4}. \quad (6)$$

$$\chi \sim N^{1/2}. \quad (7)$$

However, as pointed out in Ref. [18], the properties of the Ising model in four dimensions and the predictions of mean-field theory are not always the same. Hence, it is desirable to determine the finite size scaling behavior of various properties of the fully connected Ising model.

## II. NUMERICAL RESULTS FOR $\overline{m}$ AND $\chi$

The exact density of states  $g(M)$  of the fully connected Ising model is given by

$$g(M) = \frac{N!}{n!(N-n)!}, \quad (8)$$

where  $n = (N + M)/2$  is the number of up spins. The probability that the system has magnetization  $M$  is proportional to

$$P(M) = g(M)e^{-E/T}, \quad (9)$$

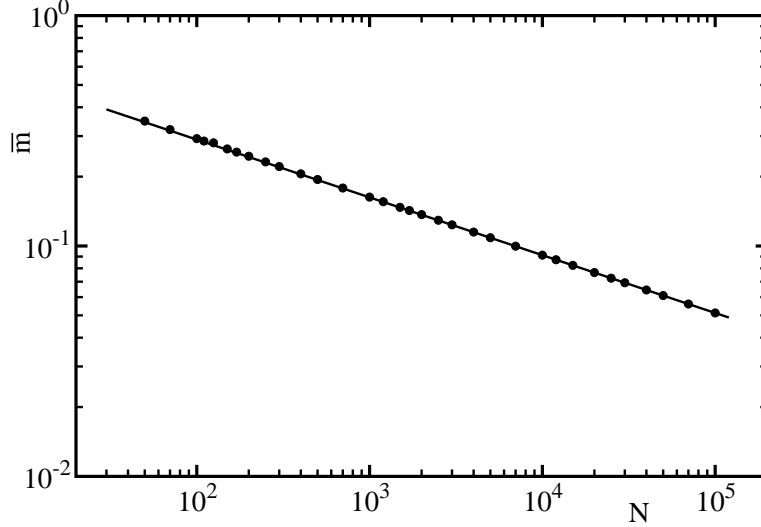


FIG. 1. Log-log plot of  $\overline{m}$ , the mean absolute magnetization per spin, versus  $N$ , the number of spins, at the critical point of the infinite fully connected Ising model,  $T_c = 4$ , for  $N \leq 10^5$  computed using the exact density of states in Eq. (8). A least squares fit to the last 10 points yields a slope of  $\approx -0.251$ , consistent with Eq. (6).

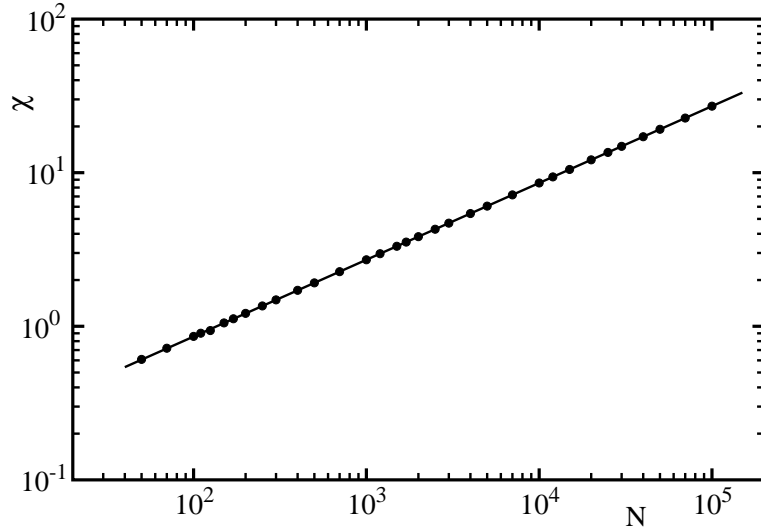


FIG. 2. Log-log plot of  $\chi$ , the susceptibility per spin, versus  $N$  at  $T = T_c$  for  $N \leq 10^5$  computed using the exact density of states in Eq. (8). The slope is  $\approx 0.500$ , consistent with Eq. (7).

with the energy  $E$  given by

$$E = \frac{J_N}{2}(N - M^2) - hM. \quad (10)$$

Note that the density of states depends only on  $M$ . We will refer to  $P(M)$  in Eq. (9) as a probability, although  $P(M)$  is not normalized.

We can evaluate  $\chi$  and  $m$  numerically as a function of  $N$  using the exact density of states

in Eq. (8). The only limitation is associated with the rapid increase of  $g(M)$  with increasing  $N$ . Our results for the  $N$  dependence of  $\bar{m}$  and  $\chi$  are shown in Figs. 1 and 2, respectively, and are consistent with Eqs. (6) and (7) for  $N \leq 10^5$ .

### III. MOST PROBABLE VALUE OF THE MAGNETIZATION

We can derive analytical expressions for the  $N$ -dependence of various quantities using the exact density of states. The usual treatment of the fully connected Ising model is based on determining the value of  $M$  that maximizes  $P(M)$ . If we use the weak form of Stirling's approximation,  $\ln x! \approx x \ln x - x$ , we find for large  $N$  that

$$\frac{d \ln P(M)}{dM} \approx \frac{1}{2} \frac{\ln(N-n)}{n} + \beta(qJM + h) = 0. \quad (11)$$

Equation (11) yields the usual mean-field result  $m = \tanh \beta(qJm + h)$ .

To find the  $N$ -dependence of  $M$  at  $T = T_c$  we use the stronger form of Stirling's approximation,  $\ln x! \approx x \ln x - x + \ln \sqrt{2\pi x}$ , so that  $d \ln x! / dx \approx \ln x + 1/2x$ . In this approximation we obtain

$$\frac{d \ln P(M)}{dM} \approx \frac{1}{2} \ln \frac{1-m}{1+m} + \frac{m}{N} \frac{1}{1-m^2} + \frac{\beta q J m}{1-1/N} + \beta h = 0. \quad (12)$$

We let  $h = 0$  and keep terms to first-order in  $1/N$  and third-order in  $m$ . The result is

$$-m - \frac{m^3}{3} + \frac{m}{N}(1+m^2) + \beta q J m \left(1 + \frac{1}{N}\right) = 0. \quad (13)$$

For  $\beta q J = 1$  ( $T = T_c$ ), several terms cancel, and we obtain [19]

$$m^2 \sim \frac{6}{N} \quad (N \gg 1). \quad (14)$$

We see from Eq. (14) that  $m \sim N^{-1/2}$ , in apparent contradiction with Eq. (6). However, the variable  $m$  in Eq. (14) is the most probable value of the magnetization rather than its mean value. Hence, the mean value and the most probable value of the magnetization scale differently with  $N$  at the critical temperature — behavior that is inconsistent with our usual understanding of mean-field.

In Fig. 3 we plot the  $N$ -dependence of  $\tilde{m}$ , the most probable (positive) value of  $m$ , as determined numerically from Eqs. (8) and (9). We see that the  $N$ -dependence of  $\tilde{m}$  is consistent with

$$\tilde{m} \sim N^{-1/2} \quad (\text{most probable value}). \quad (15)$$

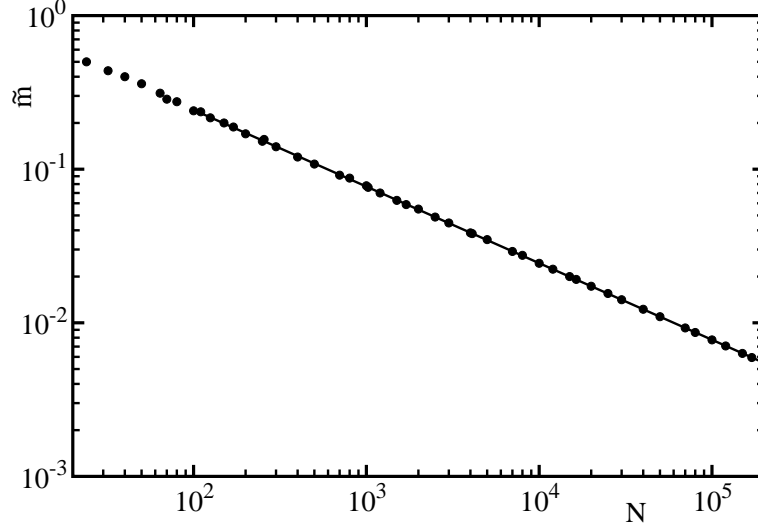


FIG. 3. The  $N$ -dependence of  $\tilde{m}$ , the most probable value of  $m$ , as determined from Eqs. (8) and (9). The slope of the log-log plot is  $-0.504$ , consistent with the analytical result in Eq. (14).

#### IV. THE PROBABILITY DISTRIBUTION

A plot of  $P(m)$  for  $N = 100$  and  $N = 800$  as determined from Eqs. (8) and (9) at  $T = T_c$  is given in Fig. 4. Note that  $P(m)$  is not a Gaussian and the maxima of  $P(m)$  are at  $|m| > 0$ , which implies that the effective critical temperature for finite  $N$  is less than the critical temperature of the infinite system.

To emphasize that the behavior of the fully connected Ising model at  $T = T_c$  is qualitatively different than at other temperatures, we plot  $P(m)$  for  $T = 3$ ,  $T = T_c$ , and  $T = 5$  in Fig. 5. As expected,  $P(m)$  has a single maximum for  $T > T_c$  and has two maxima at  $m \neq 0$  for  $T < T_c$ .

One way to characterize  $P(m)$  is to compute the reduced fourth-order (Binder) cumulant, which is defined as [20]

$$U_4 = 1 - \frac{\overline{m^4}}{3\overline{m^2}^2}. \quad (16)$$

We use Eqs. (8) and (9) to compute  $U_4$  and find that, as expected,  $U_4 \approx 0$  for  $T = 5$ , and hence  $P(m)$  is well approximated by a single Gaussian for  $T > T_c$  and large  $N$ . Similarly, for  $T = 3$  we find that  $U_4 \approx 2/3$ , which implies that  $P(m)$  is well approximated as a sum of two Gaussians [21]. We also find that  $U_4 \approx 0.276$  at  $T = T_c$  for  $N = 10^4$ , and hence  $P(m)$  is not well approximated by a Gaussian, even for large  $N$ .

It is interesting to compare the behavior of  $\tilde{m}$  and  $P(m)$  for the fully connected Ising

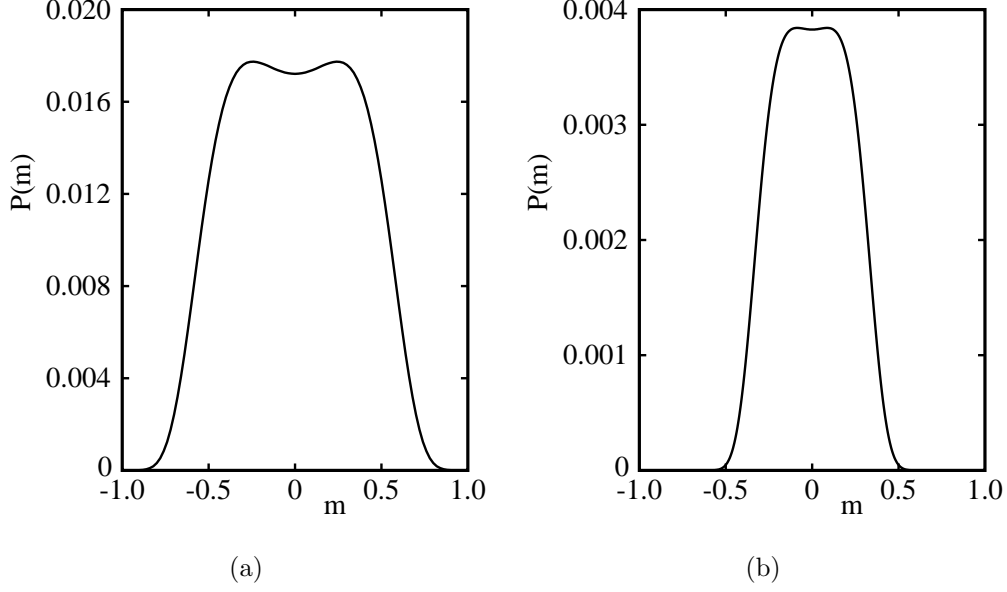


FIG. 4. Plot of  $P(m)$  versus  $m$  as determined from Eqs. (8) and (9) at  $T = T_c$  for (a)  $N = 100$  and (b)  $N = 800$ . Note that  $P(m)$  is symmetrical about  $m = 0$ , and the maxima of  $P(m)$  are at  $|m| > 0$ . It is clear that  $P(m)$  cannot be approximately by a Gaussian.

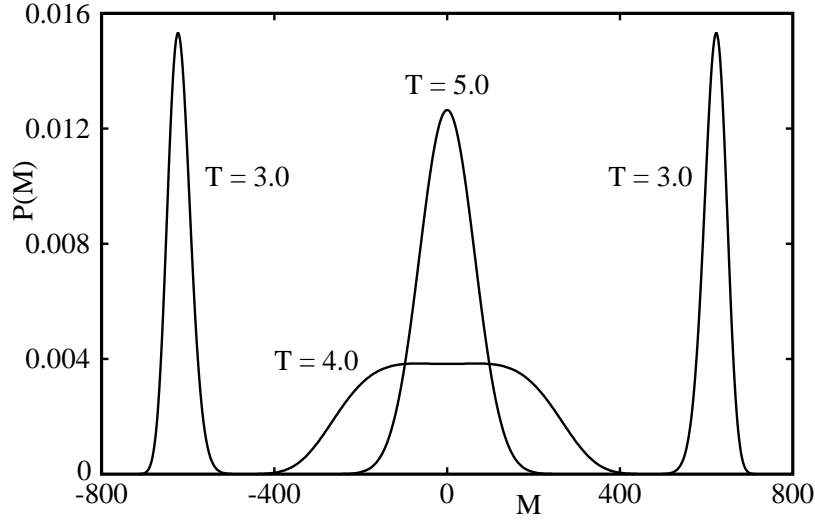


FIG. 5. Plot of  $P(M)$  at  $T = 5$ ,  $T = T_c = 4$ , and  $T = 3$  for the fully connected Ising model with  $N = 800$ . As expected,  $P(M)$  has a single maximum for  $T > T_c$  and two maxima at  $M \neq 0$  for  $T < T_c$ .

model to their behavior in the nearest-neighbor Ising model at the critical temperature of the infinite system. The maxima of  $P(m)$  for the nearest-neighbor Ising model at  $T = T_c = 2/\ln(1 + \sqrt{2})$  as obtained by a Monte Carlo simulation are also not at  $m = 0$  [22] [see Fig. 6(a)]. However, the most probable and mean values of the magnetization both scale as

$L^{-1/8}$  in two dimensions [see Fig. 6(b)], in contrast to their different scaling behavior in the fully connected Ising model.

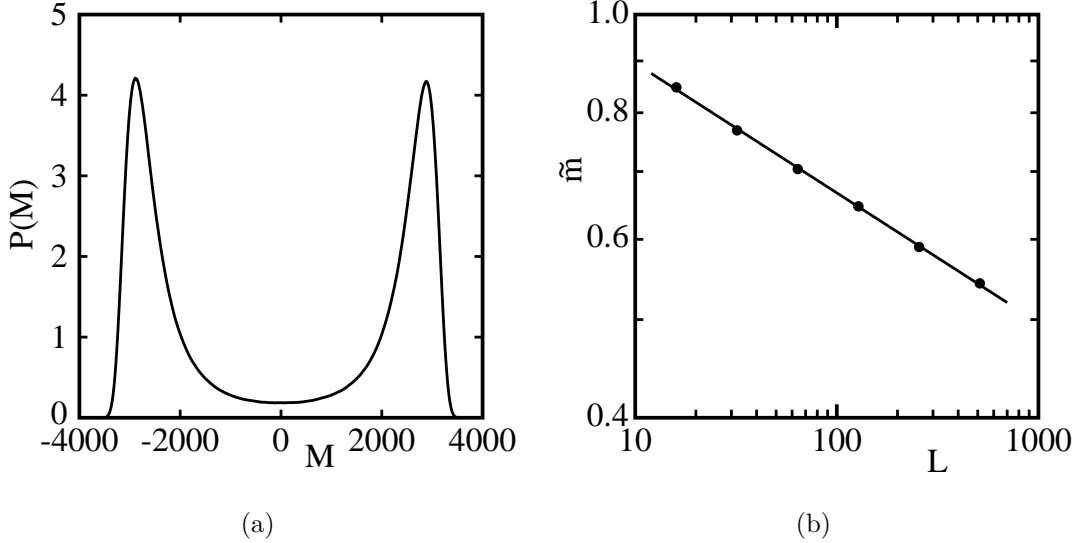


FIG. 6. Monte Carlo results for the nearest-neighbor Ising model at the critical temperature of the infinite system,  $T_c = 2/\ln(1 + \sqrt{2})$ . (a) The probability  $P(M)$  for linear dimension  $L = 64$  and  $10^8$  Monte Carlo steps per spin. Note the existence of two peaks in  $P(M)$  at  $M \approx \pm 2872$ , which implies that the effective critical temperature is less than the critical temperature of the infinite system. (b) Log-log plot of the maxima of  $P(m)$  for  $m > 0$  as a function of  $L$ . The slope is approximately 0.128, consistent with  $\beta = 0.125$ , the critical exponent for the mean magnetization.

## V. THE GINZBURG PARAMETER AND THE RESTORATION OF HYPER-SCALING

The different scaling behavior of the mean magnetization and the most probable magnetization in the fully connected Ising model implies that hyperscaling, which assumes that there is only one important scaling variable near the critical point, does not hold. As discussed in Ref. 2, hyperscaling is not satisfied by mean-field theories, but hyperscaling is restored if the Ginzburg parameter is held constant as the critical point is approached [2, 23].

The definition of the Ginzburg parameter,  $G$ , follows from the well known Ginzburg criterion for the applicability of mean-field theory [24]. This criterion requires that the fluctuations of the order parameter be small compared to its mean value, that is,  $G^{-1} = \xi^d \chi / \xi^{2d} \bar{m}^2 \ll 1$ , where  $\xi$  is the correlation length and  $d$  is the spatial dimension. In the limit



$G \rightarrow \infty$  the system is described exactly by mean-field theory. The system is near-mean-field for  $G \gg 1$  but finite.

To determine the dependence of  $G$  on  $N$  and  $\epsilon = (|T - T_c|)/T_c$ , we use the mean-field critical exponents for  $\beta$  and  $\gamma$  in Eq. (5) and obtain  $G = \xi^d \epsilon^2$  [2]. Because  $N \sim \xi^d$ , the Ginzburg parameter for the fully connected Ising model is given by (up to a numerical constant)

$$G = N\epsilon^2. \quad (17)$$

If  $G$  is held constant, we have from Eq. (17) that

$$\epsilon = \pm \left(\frac{G}{N}\right)^{1/2}. \quad (18)$$

We can show analytically that  $\tilde{m}$  scales as  $N^{-1/4}$  if  $G$  is held constant. We substitute  $T = T_c(1 + \epsilon)$  in Eq. (13), assume that  $\epsilon = -(G/N)^{1/2}$  with  $G$  a constant and  $T < T_c$ , and rewrite Eq. (13) to leading order in  $1/N$  as

$$-m - \frac{m^3}{3} + \frac{m}{1 - (G/N)^{1/2}} = 0, \quad (19)$$

where  $qJ/T_c = 1$ . If we let  $[1 - (G/N)^{1/2}]^{-1} \approx 1 + (G/N)^{1/2}$ , we obtain

$$\tilde{m} = 3^{1/2} \left(\frac{G}{N}\right)^{1/4} \sim N^{-1/4}. \quad (\text{constant Ginzburg parameter}) \quad (20)$$

The  $N^{-1/4}$ -dependence of  $\tilde{m}$  in Eq. (20) can be confirmed numerically by computing the most probable value of  $M$  from  $P(M)$  using the exact density of states and varying  $T$  as  $N$  is increased with  $G$  held constant. The results are shown in Fig. 7 and are consistent with  $\tilde{m} \sim N^{-1/4}$ .

Note that  $T_c(N)$ , the effective critical temperature for finite  $N$ , approaches the critical temperature of the infinite fully connected Ising model in such a way that the Ginzburg parameter is a constant for sufficiently large  $N$ . If we define  $T_c(N)$  as the temperature at which  $\chi$  is a maximum for a given value of  $N$ , we find that  $T_c(N)$  approaches  $T_c = 4$  with  $G = N\epsilon^2 \approx 1.53$  for  $N \gtrsim 10^4$ , where  $\epsilon = |T_c(N) - T_c|/T_c$  (see Fig. 8). Because the definition of  $G$  in Eq. (17) does not include an unknown numerical factor, the actual value of  $G = 1.53$  is not significant.

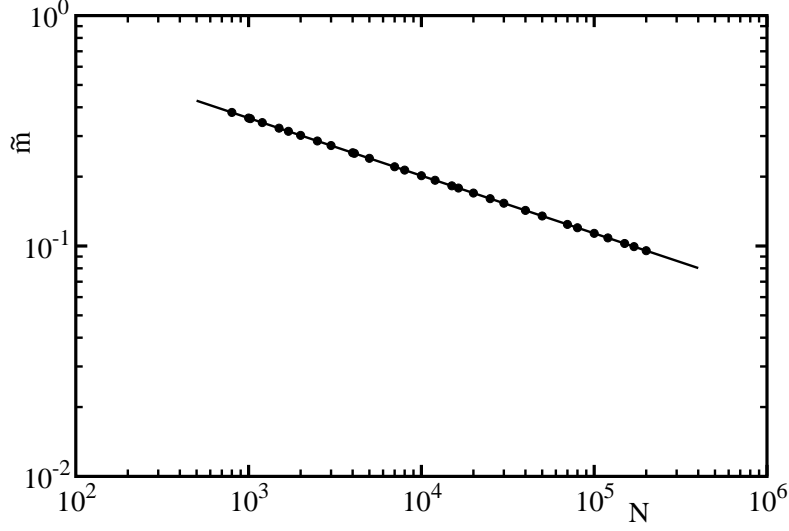


FIG. 7. Log-log plot of  $\tilde{m}$  versus  $N$  computed using the exact density states with the Ginzburg parameter held fixed at  $G = 1.53$  (the value of  $G$  is unimportant). The slope is  $-0.25$ , consistent with  $\tilde{m} \sim N^{-1/4}$  in Eq. (20).

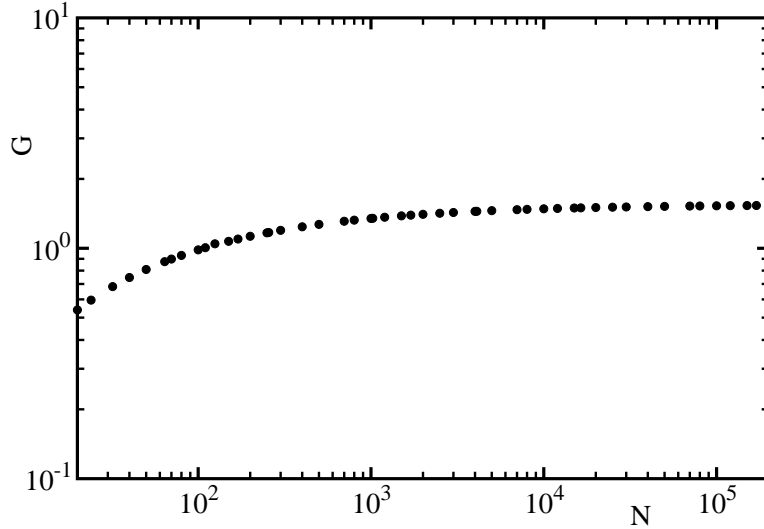


FIG. 8. The  $N$ -dependence of the Ginzburg parameter  $G = N\epsilon^2$  with  $\epsilon = [T_c(N) - T_c]/T_c$  and  $T_c(N)$  the effective critical temperature as determined from the maximum of  $\chi$ . Note that  $G$  approaches a constant for  $N \gtrsim 10^4$ .

## VI. ANALYTICAL CALCULATION OF THE MEAN MAGNETIZATION

To calculate the scaling behavior of  $\overline{m}$ , we expand  $\ln P(m)$  in a Taylor series in  $m - \tilde{m}$ , where  $\tilde{m}$  is the most probable value of  $m$  as given by Eq. (14). We have

$$\begin{aligned} \ln P(m) \approx & \ln P(\tilde{m}) + \frac{1}{2}(m - \tilde{m})^2 \frac{d^2 \ln P(m)}{dm^2} \Big|_{m=\tilde{m}} \\ & + \frac{1}{3!}(m - \tilde{m})^3 \frac{d^3 \ln P(m)}{dm^3} \Big|_{m=\tilde{m}} + \frac{1}{4!}(m - \tilde{m})^4 \frac{d^4 \ln P(m)}{dm^4} \Big|_{m=\tilde{m}}. \end{aligned} \quad (21)$$

In analogy to the form of the free energy in Landau-Ginzburg theory, we will need to keep terms only to fourth-order in  $(m - \tilde{m})^4$  [25]. We also expect that the second and third derivatives of  $\ln P(m)$  to both approach zero as  $N \rightarrow \infty$  and  $(d^4 \ln P(m)/dm^4)_{m=\tilde{m}}$  to be independent of  $N$ .

We have to leading order in  $1/N$  that

$$\frac{d^2 \ln P(m)}{dm^2} = -\frac{1}{1-m^2} + \frac{\beta q J}{1-1/N} + \frac{1}{N} \frac{1+m^2}{(1-m^2)^2}, \quad (22)$$

and hence

$$\frac{d^2 \ln P}{dm^2} \Big|_{m=\tilde{m}, T=T_c} \approx -1 - \tilde{m}^2 + 1 + \frac{1}{N} + \frac{1}{N} = -\frac{4}{N}. \quad (23)$$

Note that  $(d^2 \ln P/dm^2)_{m=\tilde{m}} < 0$ , which is consistent with  $\tilde{m}$  being the most probable value.

We also have to leading order that

$$\frac{d^3 \ln P}{dm^3} = -\frac{2m}{(1-m^2)^2} \text{ and } \frac{d^4 \ln P}{dm^4} = -\frac{2}{(1-m^2)^2}. \quad (24)$$

Hence to leading order in  $1/N$  we have

$$\frac{d^3 \ln P}{dm^3} \Big|_{m=\tilde{m}} \approx -2 \left( \frac{6}{N} \right)^{1/2} \text{ and } \frac{d^4 \ln P}{dm^4} \Big|_{m=\tilde{m}} \approx -2. \quad (25)$$

We can interpret  $\ln P(m)$  as the free energy per spin. Because  $(d^2 \ln P(m)/dm^2)_{m=\tilde{m}}$  and  $(d^3 \ln P(m)/dm^3)_{m=\tilde{m}}$  both go to zero as  $N \rightarrow \infty$ , we have from Eqs. (21) and (25) that [11]

$$\overline{m} = \frac{\int_0^1 m e^{-N(m-\tilde{m})^4/12} dm}{\int_0^1 e^{-N(m-\tilde{m})^4/12} dm} \quad (N \gg 1). \quad (26)$$

We change variables to  $x = (m - \tilde{m})(N/12)^{1/4}$  and keep only the leading order term in  $N$ . The upper limit of integration,  $x_{\max} = (1 - \tilde{m})(N/12)^{1/4} \sim N^{1/4} \rightarrow \infty$  as  $N \rightarrow \infty$ . Similarly,

the lower limit of integration  $x_{\min} = -\tilde{m}(N/12)^{1/4} \sim N^{-1/4} \rightarrow 0$  as  $N \rightarrow \infty$ . Hence, for large  $N$  we obtain

$$\overline{m} = \left(\frac{12}{N}\right)^{1/4} \frac{\int_0^\infty x e^{-x^4} dx}{\int_0^\infty e^{-x^4} dx} \approx 0.91 N^{-1/4}. \quad (27)$$

The leading correction to  $\overline{m}$  in Eq. (27) is proportional to  $N^{-1/2}$ . Similar considerations yield the scaling behavior of  $\chi$  given in Eq. (7).

It is easy to check that  $(d^n \ln P(m)/dm^n)_{m=\tilde{m}}$  for  $n > 4$  is either independent of  $N$  ( $n$  even) or proportional to  $N^{-1/2}$  ( $n$  odd), thus justifying the assumption in Eq. (21) that higher-order terms in the expansion of  $\ln P(m)$  can be neglected.

The form of  $\ln P(m)$  in Eq. (21) can be used to compute the cumulant defined in Eq. (16). The result is  $U_4 \approx 0.271$  at  $T = T_c$ , which is consistent with the computed value of  $U_4 = 0.276$  using the exact density of states for  $N = 10^4$ .

## VII. SCALING AT THE SPINODAL

### A. Simple scaling argument

Because the spinodal is a line of critical points, we expect that finite size scaling at the Ising spinodal proceeds similarly to our analysis at the Ising mean-field critical point. We assume that  $T < T_c$  and vary the field  $h$  near the spinodal field  $h_s$ . In terms of  $\Delta h = (h - h_s)/h_s$  the usual scaling relations are [2]

$$\overline{\psi} \sim \Delta h^{1/2} \quad (28)$$

$$\chi \sim \Delta h^{-1/2} \quad (29)$$

$$\xi \sim \Delta h^{-1/4}, \quad (30)$$

where the order parameter  $\overline{\psi} = \overline{m} - m_s$  is related to the mean magnetization per spin near the spinodal, and  $m_s$  is the value of the magnetization at the spinodal. We use Eq. (30) to obtain  $\overline{\psi} \sim \xi^{-2}$  and  $\chi \sim \xi^2$ . If we take the upper critical dimension to be six at the spinodal [26], we have  $N \sim \xi^6$ , and hence

$$\overline{\psi} \sim N^{-1/3} \quad (31)$$

$$\chi \sim N^{1/3}. \quad (32)$$

## B. Numerical results

The numerical evaluation of the  $N$ -dependence of various quantities such as  $\bar{\psi}$  and  $\chi$  as a function of  $N$  at  $h = h_s$  using the exact density of states in Eq. (8) is more subtle than at the critical temperature because we must include only values of  $M$  corresponding to the metastable state. To understand this restriction, imagine a Monte Carlo simulation of the fully connected Ising model at temperature  $T < T_c$  and magnetic field  $h = h_0 > 0$ . Because  $h_0 > 0$ , the values of  $M$  are positive. After equilibrium has been reached, we let  $h \rightarrow -h_0$ . If  $h_0$  is not too large, the system will remain in a metastable state for a reasonable number of Monte Carlo steps per spin. To compute  $\chi$  associated with the pseudospinodal (the spinodal is defined only in the limit  $N \rightarrow \infty$  for the fully connected Ising model), we must include only those values of  $M$  that are representative of the metastable state. As discussed in Ref. 12, the values of  $M$  that may be included in thermal averages of the metastable state must satisfy the condition that  $M \geq M_{\text{ip}}$ , where  $M_{\text{ip}}$  is the value of  $M$  at the inflection point of  $P(M)$ . We set  $d^2 \ln P(M)/dM^2 = 0$  and use Eq. (22) to find that [12]

$$M_{\text{ip}} = \sqrt{N^2 \left(1 - \frac{1}{\beta q J}\right) + \frac{N}{\beta q J}}. \quad (33)$$

We follow Ref. [27] and choose  $T = 4T_c/9 = 16/9$ . Hence  $z = \beta q J = 9/4$  in Eq. (33). For this value of  $z$  we obtain  $h_s \approx 1.2704$  [28].

Our numerical results for  $\chi$  at  $h = h_s$  for increasing values of  $N$  are shown in Fig. 9 using the exact density of states in Eq. (8) and values of  $M > M_{\text{ip}}$ . The slope of  $\approx 0.34$  is consistent with Eq. (32). Similarly, we find that a log-log plot of  $\bar{\psi}$  versus  $N$  yields a slope of  $-0.33$  [see Fig. 10(a)] in agreement with Eq. (31). A log-log plot of the most probable values of  $m$  near the spinodal yields the scaling behavior [see Fig. 10(b)]

$$\tilde{\psi} \sim N^{-1/2}. \quad (34)$$

We see that the  $N$ -dependences of the most probable and mean values of the magnetization at the spinodal differ as they do at the critical point.

## C. Analytical derivation

The analytical calculation of the  $N$ -dependence of  $\bar{\psi}$ ,  $\tilde{\psi}$ , and  $\chi$  at the spinodal proceeds similarly to the derivation at the critical temperature. We can use Eq. (12) with  $N \rightarrow \infty$

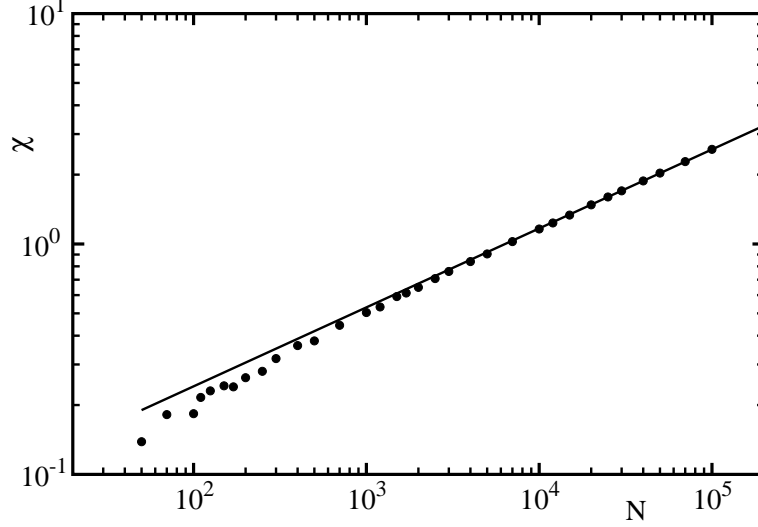


FIG. 9. Log-log plot of  $\chi$ , the susceptibility per spin, versus  $N$  at  $h = h_s$  and  $T = 16/9$  for  $N \leq 2 \times 10^5$  computed using the exact density of states in Eq. (8) and the requirement that  $M \geq M_{\text{ip}}$ . The slope is  $\gamma_n \approx 0.34$  from a least squares fit to the last seven values of  $\chi$ , consistent with the exponent  $1/3$  in Eq. (32).

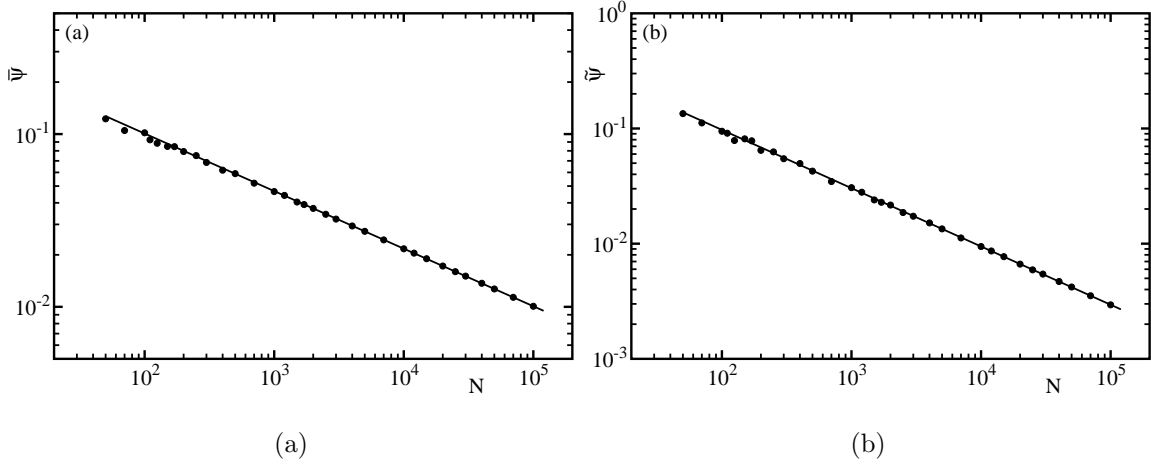


FIG. 10. (a) Log-log plot of  $\bar{\psi}$  versus  $N$  at  $h = h_s$  and  $T = 4T_c/9$ . The slope is  $\approx -0.33$ , which is consistent with Eq. (31). (b) The log-log plot of the most probable value of  $\tilde{\psi}$  at  $h = h_s$  and  $T = 4T_c/9$  gives a slope of  $\approx -0.51$ , consistent with the exponent in Eq. (34).

to show that the value of  $m$  at the spinodal is given by  $1/(1 - m_s^2) - q\beta J = 0$ , or

$$m_s = \sqrt{\frac{\beta q J - 1}{\beta q J}} = \sqrt{\frac{z - 1}{z}}. \quad (35)$$

The corresponding value of  $h_s$  can be obtained by substituting  $m = m_s$  into Eq. (12) in the limit  $N \rightarrow \infty$ .

To find the leading correction to the most probable value of  $m$  near the spinodal, we substitute  $m = m_s + \psi$  in Eq. (12) and assume that  $\psi \ll m_s$  for  $N \gg 1$ . The result is

$$\begin{aligned} \frac{d \ln P}{dm} \approx & \frac{1}{2} \ln \frac{1 - m_s}{1 + m_s} + z m_s + \beta h_s - \frac{1}{1 - m_s^2} \psi + z \psi - \frac{m_s}{(1 - m_s^2)^2} \psi^2 \\ & + \frac{1}{N} \frac{m_s}{1 - m_s^2} + \frac{1}{N} \frac{1 + m_s^2}{(1 - m_s^2)^2} \psi + \frac{z m_s}{N} + \frac{z \psi}{N} = 0. \end{aligned} \quad (36)$$

The sum of the first three terms on the right-hand side is zero. We will assume that  $\psi \sim N^{-1/2}$  and determine if this assumption is consistent with the solution to Eq. (36).

The terms proportional to  $N^{-1/2}$  are

$$\left[ -\frac{1}{1 - m_s^2} + z \right] \psi, \quad (37)$$

which sum to zero using Eq. (35). The terms proportional to  $N^{-1}$  include

$$-\frac{m_s}{(1 - m_s^2)^2} \psi^2 + \frac{1}{N} \frac{m_s}{1 - m_s^2} + \frac{z m_s}{N}, \quad (38)$$

which also must sum to zero. The result for  $\psi^2$  to order  $1/N$  is

$$\psi^2 = \frac{(1 - m_s^2)^2}{N} \left[ \frac{1}{1 - m_s^2} + z \right] = \frac{2}{N z}. \quad (39)$$

The quantity  $\psi$  in Eq. (39) represents the most probable value, which we write in the following as  $\tilde{\psi}$ . Hence, we conclude that  $\tilde{\psi} \sim N^{-1/2}$ , in agreement with the numerical result in Eq. (34).

Near the spinodal we can show that the Ginzburg parameter  $G_s$  is given by  $G_s = \xi^d \tilde{\psi}^2 / \chi \sim N \Delta h^{3/2}$ , where we have used Eqs. (30) and (32). In analogy to our discussion in Sec. V, we can show that  $\tilde{\psi} \sim N^{-1/3}$  if  $G_s$  is held fixed as  $\Delta h$  is varied at constant temperature.

Similarly, we find for large  $N$  that

$$\frac{d^2 \ln P}{dm^2} = -\frac{2m_s}{(1 - m_s^2)^2} \psi \sim N^{-1/2}, \quad (40)$$

and

$$\frac{d^3 \ln P}{dm^3} = -\frac{2m_s}{(1 - m_s^2)^3} \sim N^0. \quad (41)$$

We see that  $d^2 \ln P / dm^2 \sim N^{-1/2}$  and  $d^3 \ln P / dm^3$  is independent of  $N$  in the limit  $N \rightarrow \infty$ . Hence, we can show that  $\tilde{\psi} \sim N^{-1/3}$  and  $\chi \sim N^{1/3}$  at the spinodal in agreement with Eqs. (31) and (32).

## VIII. DISCUSSION

The different scaling behaviors of the mean and the most probable values of the magnetization at the critical temperature of the fully connected Ising model in the  $N \rightarrow \infty$  limit implies that the Gaussian approximation often associated with mean-field theory does not hold. Moreover, we find that the probability distribution of the magnetization is not a Gaussian, even in the limit  $N \rightarrow \infty$ . Our results are also not consistent with assuming that all scaling properties of the fully connected Ising model at the critical temperature of the infinite system are the same as the nearest-neighbor Ising model at the upper critical dimension, where hyperscaling is satisfied and the Ginzburg parameter is independent of the distance from the critical point and the spinodal. The reason for the difference is that hyperscaling is not satisfied at the critical temperature of the infinite system as  $N$  is increased. To do mean-field theory properly and to restore hyperscaling, we have to keep the Ginzburg parameter constant as  $N$  is increased.

The breakdown of hyperscaling does not affect the values of thermodynamic exponents such as  $\beta$ ,  $\gamma$ , and  $\alpha$  [29]. In contrast, the most probable value of the magnetization is not a thermodynamic quantity and is affected by the breakdown of hyperscaling.

It is remarkable that the fully connected Ising model, which is discussed in some undergraduate textbooks because of its simplicity [8], still yields surprises. In particular, the behavior of the fully connected Ising model at the critical point differs from that of the long-range Ising model with the Kac form of the interaction. This conclusion is not surprising because the interaction between spins in the fully connected Ising model does not have the Kac form for which mean-field theory has been shown to be exact if the thermodynamic limit is taken *before* the range of the interaction is taken to infinity.

Our results are a reminder that the applicability of mean-field theories is subtle. A recent example is found in Ref. [18], where it was shown that the divergence of the specific heat of the long-range Ising model in one and two dimensions is neither mean-field nor has the exponents associated with the nearest-neighbor Ising model. We also note that experiments in systems that are well approximated by mean-field theory are not usually done at fixed Ginzburg parameter. Hence, the interpretation for experimental results of such systems should be done with caution.



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